Fixing improper colorings of graphs

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Definition of the problem

For two r-colorings φ and φ' of G, we define:

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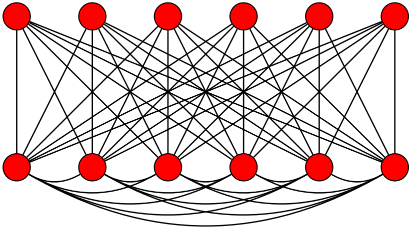
Problem: r-Color-Fixing (r-Fix)

Instance: A graph G, integer k, an r-coloring φ of $V(G)$.
Question: Does there exist a proper r-coloring φ' of G such that $\mathrm{dist}(\varphi, \varphi') \leq k$?

$$\overline{\chi}^r_\varphi(G) = \min\{\mathrm{dist}(\varphi, \varphi') : \varphi' \text{ is a proper } r\text{-coloring of } G\}$$

Example

$n = 2p$
$r = p + 1$ (equal to the chromatic number)



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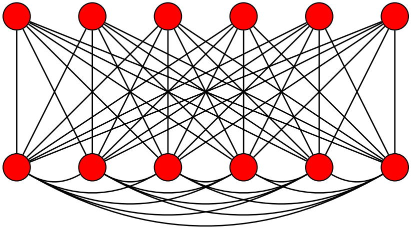

We need to recolor all vertices from K_p.

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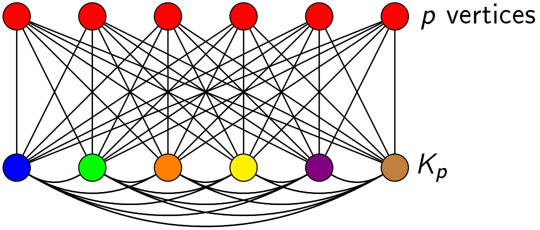

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$$\overline{\chi}_{\varphi}^{p+1}(G_p) = p$$

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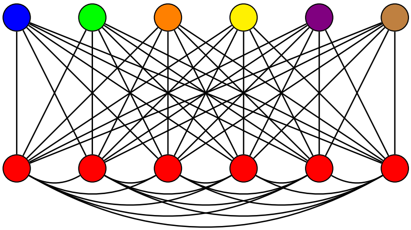

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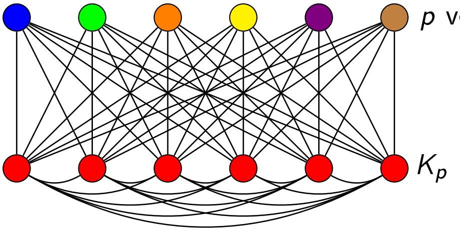

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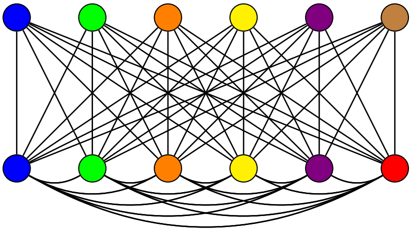

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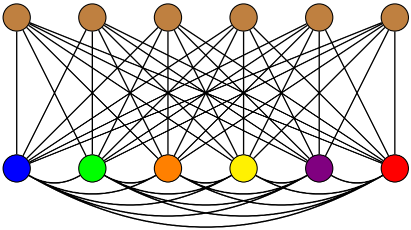

We need to recolor $p - 1$ vertices from K_p.
But then we have to recolor $p - 1$ „upper” vertices.
All but two vertices have to be recolored.

$$\overline{\chi}_{\varphi'}^{p+1}(G_p) = 2p - 2$$

Complexity of the problem: k is a part of the input

If all vertices are colored with the same color, then the r-Fix problem (for $k = n$) is equivalent to r-Colorability.

Observation

r-Fix *problem is NP-complete for any* $r \geq 3$ *(when the number* k *of allowed recoloring operations is a part of the input).*

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→ The case for $r = 1$ is trivial.

→ For $r = 2$, if G is not bipartite, then the answer is No.

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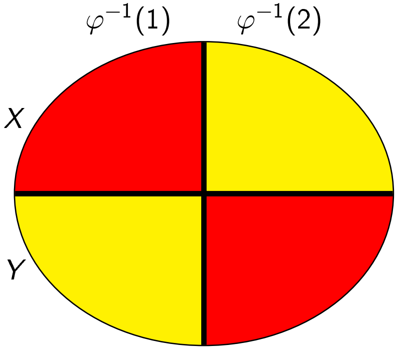

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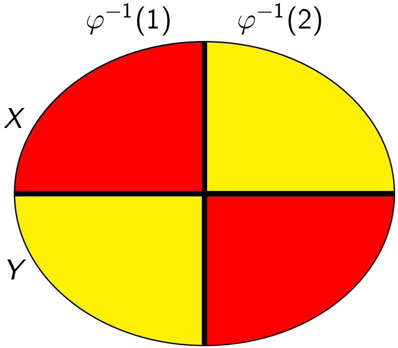

Observation

Let G be a connected bipartite graph with bipartition classes X and Y and let φ be a 2-coloring of G. Then we have

$$\overline{\chi}^2_\varphi(G) = \min\{|\left(X \ominus \varphi^{-1}(1)\right)|, |\left(X \ominus \varphi^{-1}(2)|\right)\}.$$

Complexity of the problem: k is a parameter

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Theorem

For fixed r, the r-Fix problem is in FPT, when parametrized by k.
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For fixed r, the r-Fix problem is in FPT, when parametrized by k. (there exist an algorithm with running time $f(k) \cdot poly(n)$)

This can be shown using a simple branching algorithm.

Parametrized algorithm

Algorithm: Fix(r, $\mathcal{I} = (G, k, \varphi)$)

```
if φ is a proper coloring of G then return Yes
if k = 0 then return No
xy ← any edge such that φ(x) = φ(y)
foreach v ∈ {x, y} do
    foreach col ∈ [r] \ {φ(v)} do
        φ' ← φ with vertex v recolored to col
        if Fix(r, (G, k − 1, φ')) = Yes then return Yes
return No
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The algorithm *Fix* solves the r-Fix problem in time

$$T(n, k) \leq (2(r-1))^k \cdot n^{\mathcal{O}(1)}.$$

Exact exponential algorithm

The r-Fix problem can be reduced to:

Min Weighted Partition Problem

Instance: A set N, integer d and functions $f_1, f_2, \ldots, f_d : 2^N \to [-M, M]$ for some integer M.
Question: What is the minimum w, for which there exists a partition $S_1, S_2, \ldots, S_d$, such that $\sum_{i=1}^{d} f_i(S_i) = w$?

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Reduction

$N = V(G)$, $d = r$, $M = r \cdot n$,

$$f_i(S) = \begin{cases} |S \setminus \varphi^{-1}(i)| & \text{if } S \text{ is independent,} \\ r \cdot n & \text{otherwise.} \end{cases}$$

Exact exponential algorithm

Theorem (Björklund, Husfeldt and Koivisto)

The Min Weighted Partition *problem can be solved in time $\mathcal{O}^*(2^n d^2 M)$ using exponential space and in time $\mathcal{O}^*(3^n d^2 M)$ using polynomial space.*

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Corollary

The r-Fix *problem for any fixed r can be solved in time $\mathcal{O}^*(2^n)$ using exponential space and in time $\mathcal{O}^*(3^n)$ using polynomial space.*

If G is a tree (or, more generally, a graph with bounded treewidth), we can use a standard dynamic programming approach:

$K[v,i]$ – the minimum number of vertices, which need to be recolored to obtain a proper coloring of the subtree rooted at a vertex v, such that v gets color i.

$$K[v,i] = \begin{cases} [\varphi(v) \neq i] & v \text{ is a leaf,} \\ [\varphi(v) \neq i] + \sum_{u \in children(v)} \min_{j \neq i} K[u,j] & \text{otherwise.} \end{cases}$$

Graphs with bounded treewidth

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Theorem

*For any fixed r, the optimization version of r-*Fix* problem can be solved in time $\mathcal{O}(n \cdot r^{t+2})$, where n is the number of vertices of the input graph and t is its treewidth.*

Fixing number

A *fixing number* of a graph G is the maximum number of vertices needed to be recolored to obtain a proper coloring from any coloring of G with at least $\chi(G)$ colors, i.e.

$$\overline{\chi}(G) = \max\left\{\overline{\chi}^r_\varphi(G) : \varphi\colon V(G) \to [r], r \geq \chi(G)\right\}.$$

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- This bound is tight e.g. for complete graphs.
- $\overline{\chi}(C_{2k+1}) = k$ (compared to roughly $\frac{4k}{3}$ given by the theorem above).

Further research

Open problem

What is the complexity of the problem for $r = 4$ and G planar?

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What is the complexity of the problem if r is not fixed?

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Find a polynomial kernel for r-Fix (parametrized by k).

Open problem

- *Find better bounds for $\overline{\chi}(G)$ for planar G (or for some other reasonable class of graphs).*
- *Find another classes of graphs in which the general bound can be beaten.*