Towards a Characterization of Leaf Powers by Clique Arrangements

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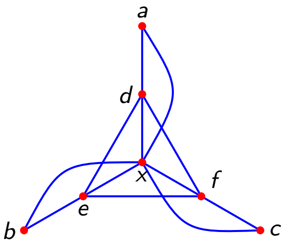
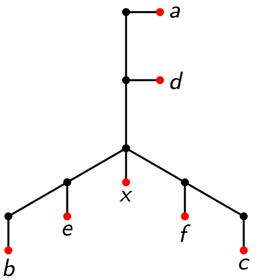
k-Leaf Powers (Nishimura et al. 2002)

All graphs $G = (V, E)$ with a k-leaf root, that is, a tree T with V as the set of leaves such that $xy \in E$ if and only if $\delta_T(x, y) \leq k$.



$k = 4$



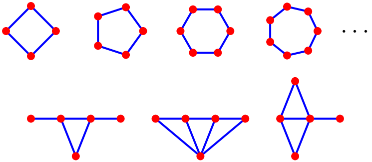
2-Leaf Powers

- disjoint union of cliques
- forbidden subgraphs: 
- straight forward recognition in linear time

3-Leaf Powers

(Brandstädt, Le 2006)

- trees with cliques substituted into vertices
- forbidden subgraphs:

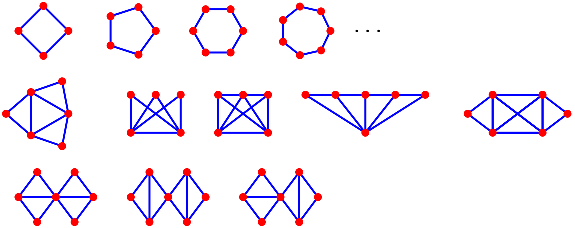


- linear time recognition by *peo*

4-Leaf Powers

(Brandstädt, Le, Sritharan 2008)

- basic 4-leaf powers with cliques substituted into vertices
- forbidden subgraphs of basic 4-leaf powers:



- forbidden subgraphs of 4-leaf powers unknown
- rather complex linear time recognition using structure of blocks

5-Leaf Powers

(Brandstädt, Le, Rautenbach 2009)

- characterization still open
- complex linear time recognition (Chang, Ko 2007)
- distance hereditary 5-leaf powers are gained by substituting cliques into 3-leaf powers, plump darts, or plump bulls (that are, basic distance hereditary 5-leaf powers)
- already 34 forbidden subgraphs for basic distance hereditary 5-leaf powers

k-Leaf Powers for $k \geq 6$?

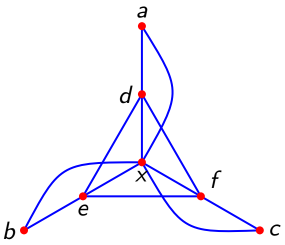
- No essential progress for years now!
- What happens, if we push k to infinity?

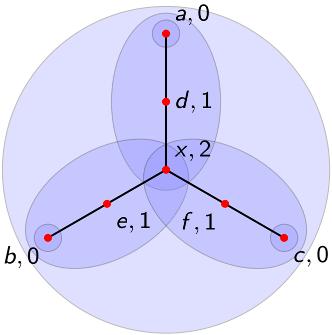
Leaf Powers

All graphs that are a k-leaf power for some $k \in \mathbb{N}$.

Intersection Model (Brandstädt, Hundt, Mancini, Wagner 2009)

A graph $G = (V, E)$ is a leaf power if and only if is the intersection graph of neighborhood subtrees in a tree.



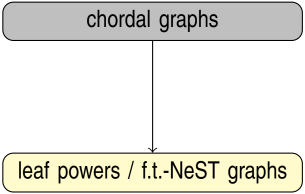


Classification of Leaf Powers

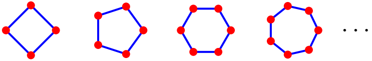
leaf powers / f.t.-NeST graphs

- leaf powers are equivalent to fixed tolerance neighborhood subtree tolerance (ft-NeST) graphs by Brandstädt, Hundt, Mancini, Wagner 2009
- likewise, recognition and characterization are open

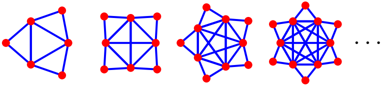
Classification of Leaf Powers



- every cycle of length at least four has chord
- superset of leaf powers as induced subgraphs of tree powers are cycle free
- intersection graphs of subtrees in trees
- forbidden subgraphs: induced cycles of length at least four



- beyond leaf powers: k-suns, $k \geq 3$



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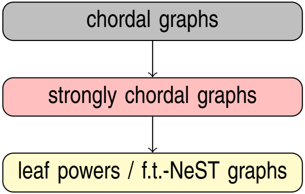
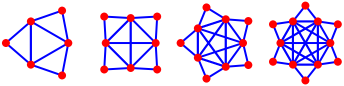
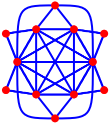


- every even cycle has odd chord
- superset of leaf powers as induced subgraphs of tree powers are sun free
- additionally forbidden: k-suns, $k \geq 3$



...

- beyond leaf powers: (Bibelnieks, Dearing 1993)



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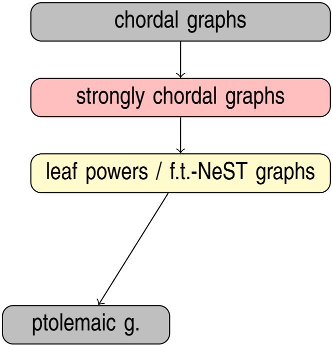
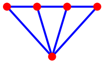
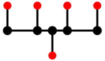


- fulfill ptolemaic's inequality
- subset of leaf powers by Brandstädt, Hundt 2008
- additionally forbidden: gem



- proper subset as gem is 4-leaf power



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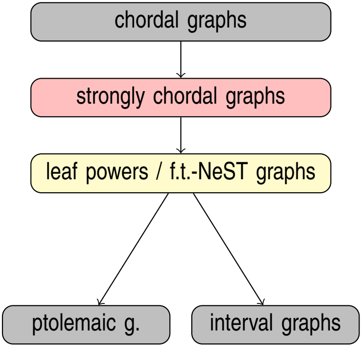
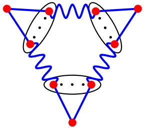


- intersection graphs of line segments on the real line
- subset of leaf powers by Brandstädt, Hundt 2008
- additionally forbidden: asteroidal triples



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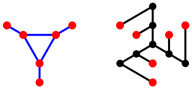

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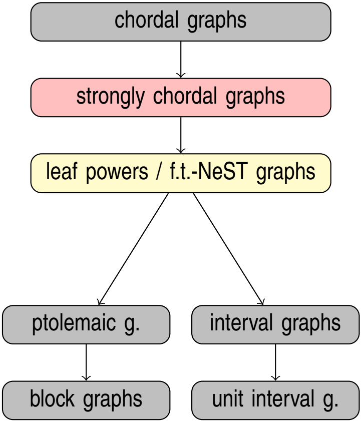
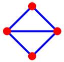
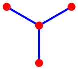


- all 2-connected components (blocks) are cliques
- additionally forbidden: diamond



- intersection graphs of unit length line segments on the real line
- additionally forbidden: claw



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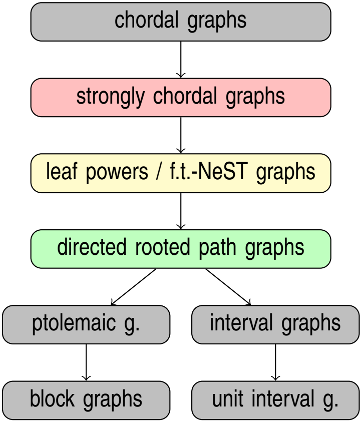
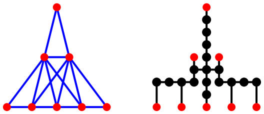


- intersection graphs of directed paths in rooted directed trees
- subset of leaf powers by Brandstädt, Hundt, Mancini, Wagner 2009
- characterization unknown
- proper subset as planet is 7-leaf power



Tools for (Strongly) Chordal Graphs

- clique tree
- (weighted) clique graph
- clique separator graph
- many others

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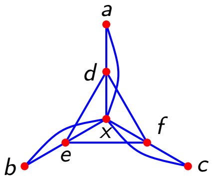

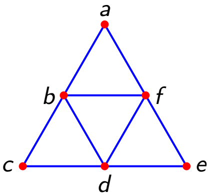

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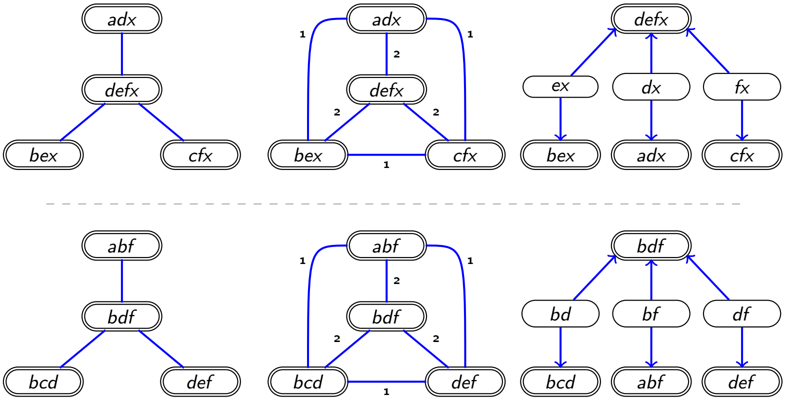
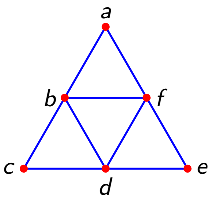


New: The Clique Arrangement



Clique Arrangement (Nevries, Rosenke 2013)

For a chordal graph G the directed graph $\mathcal{A}(G) = (\mathcal{X}, \mathcal{E})$ is given by

- $\mathcal{X} = \{\bigcap_{C \in \mathcal{C}} C \mid \mathcal{C}$ is subset of the maximal cliques of $G\}$ and
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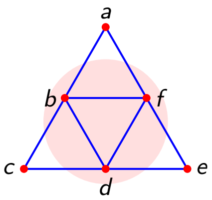

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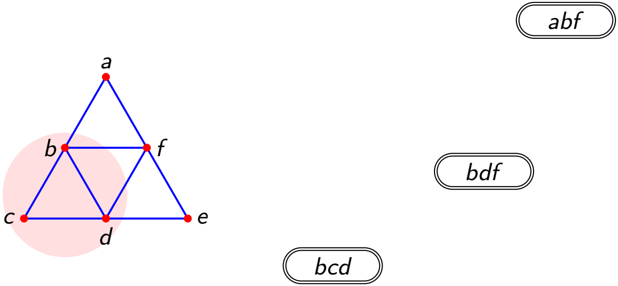

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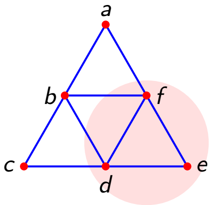

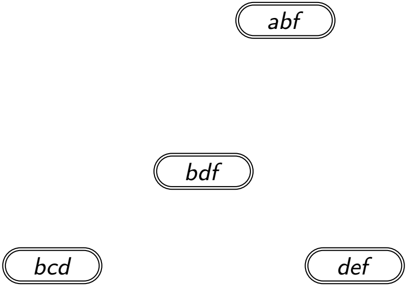

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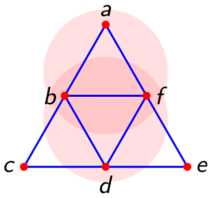

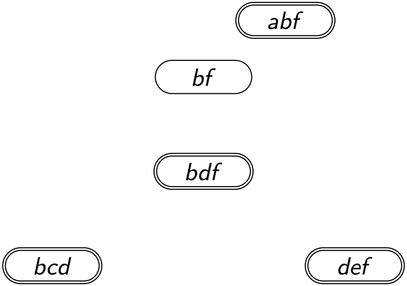

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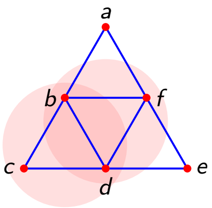

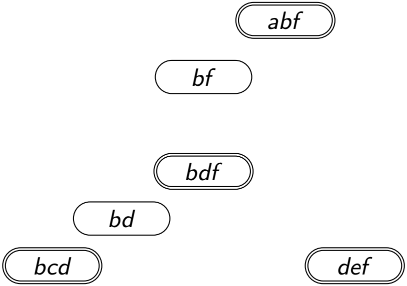

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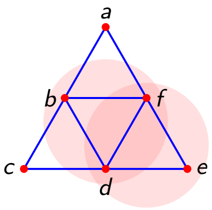

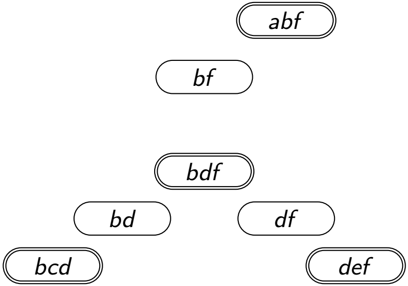

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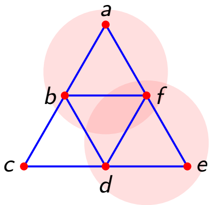

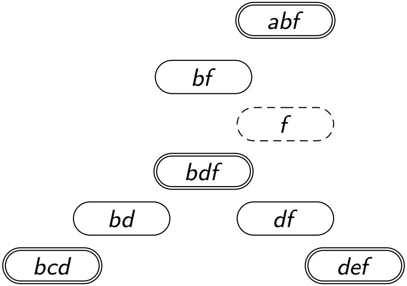

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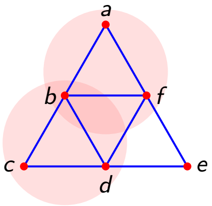

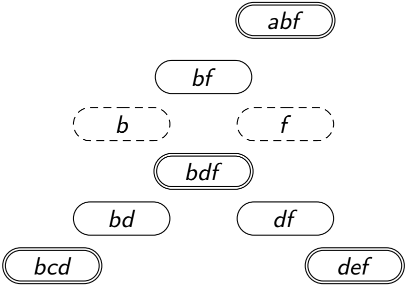

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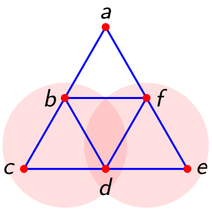

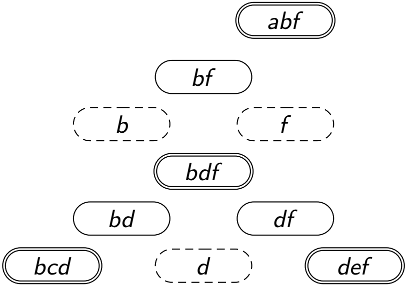

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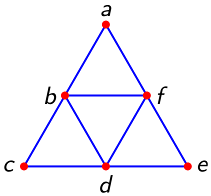

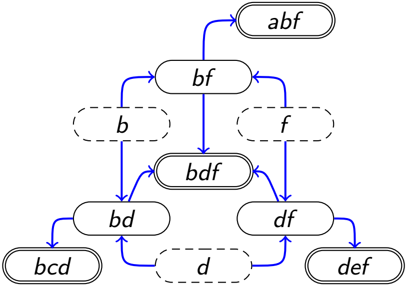

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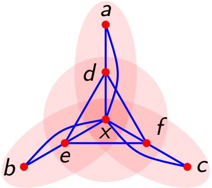

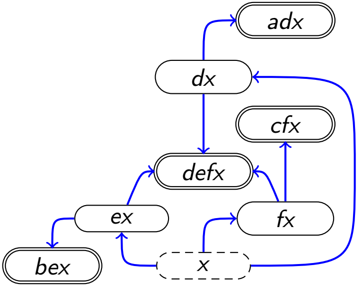

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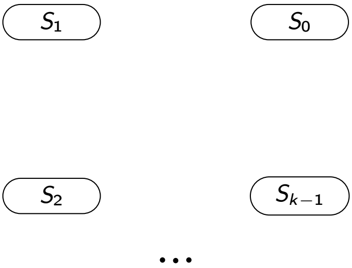
Strong Chordality in Clique Arrangements

Bad k-Cycle, $k \geq 3$

- contain start nodes $S_0, \ldots, S_{k-1}$
- and terminal nodes $T_0, \ldots, T_{k-1}$,
- such that S_i reaches $T_j \Leftrightarrow j = i$ or $j = i - 1 \mod k$.



Theorem (Rosenke et al. 2013)

A graph is strongly chordal $\Leftrightarrow$ the clique arrangement is free of bad k-cycles for $k \geq 3$.

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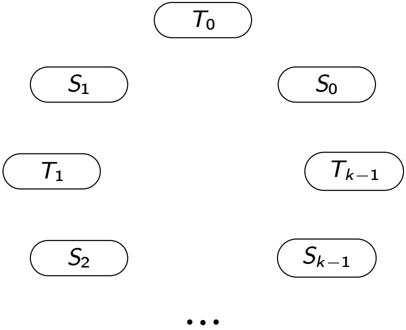
A graph is strongly chordal $\Leftrightarrow$ the clique arrangement is free of bad k-cycles for $k \geq 3$.



Strong Chordality in Clique Arrangements

Bad k-Cycle, $k \geq 3$

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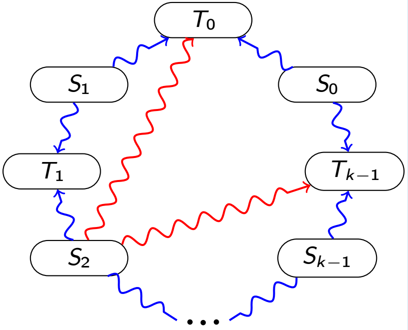

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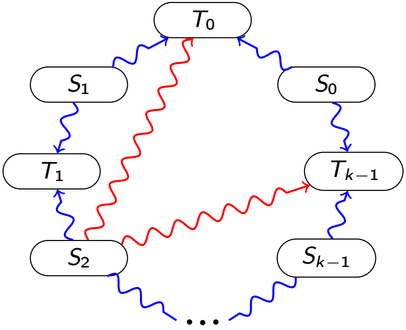

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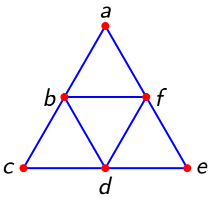
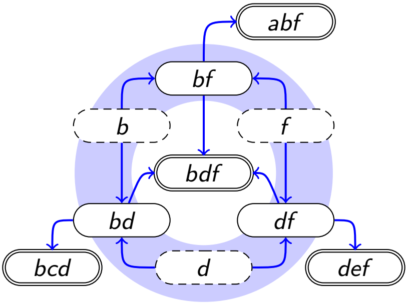
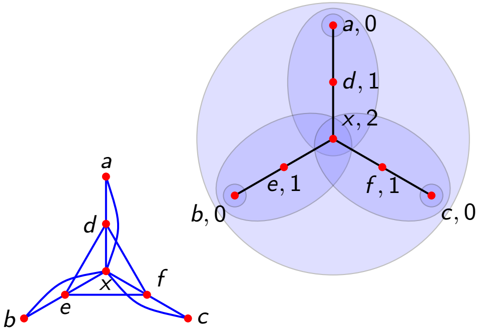




Clique Arrangements and Leaf-Roots



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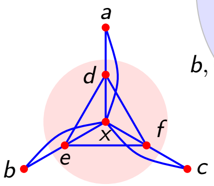

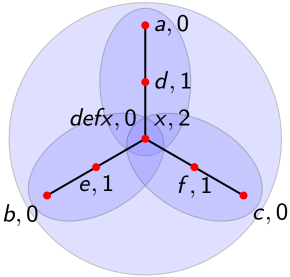

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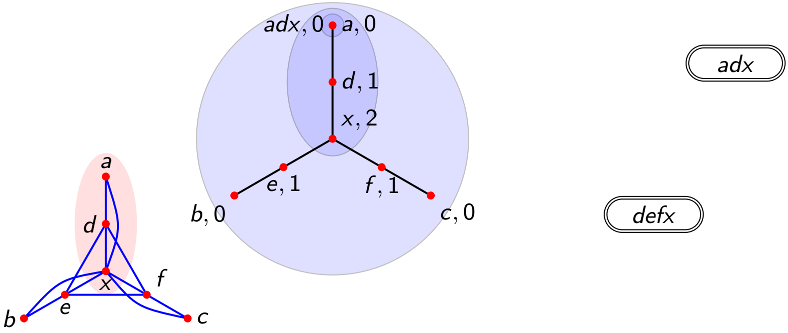

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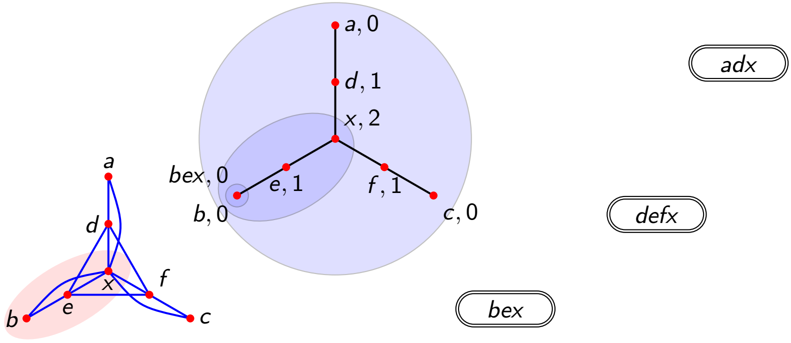

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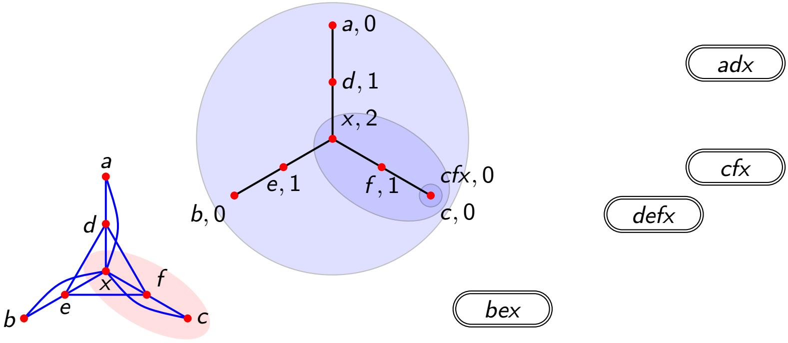

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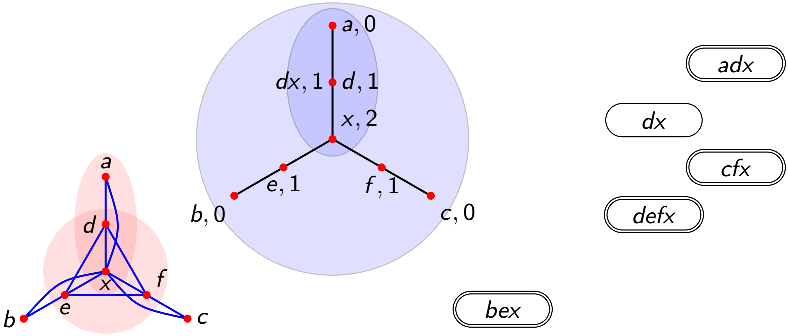

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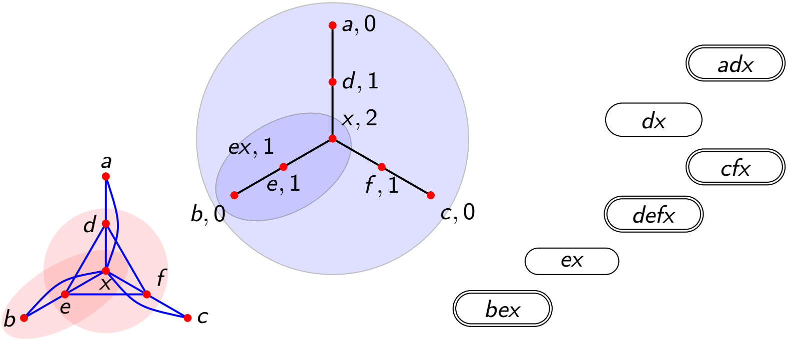

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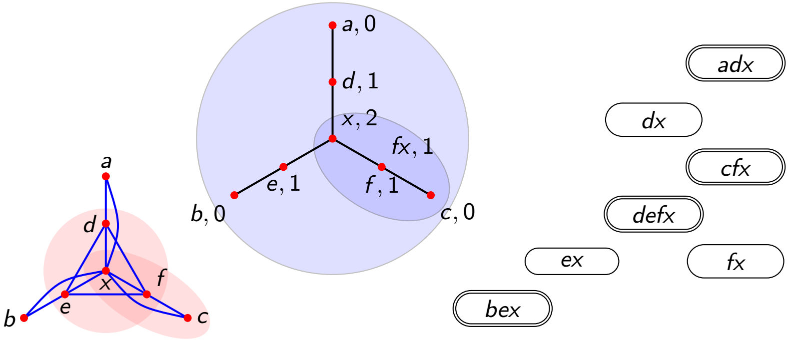

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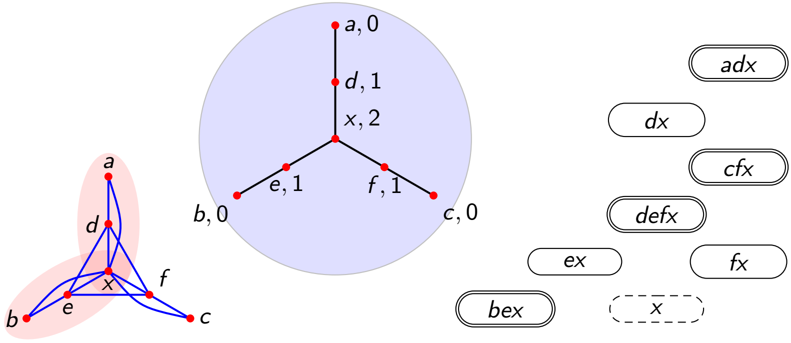

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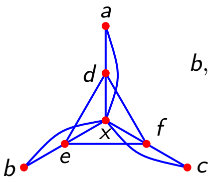

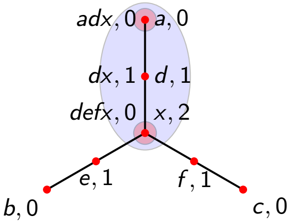

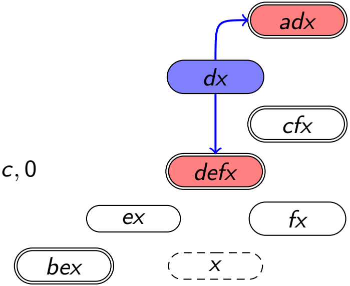

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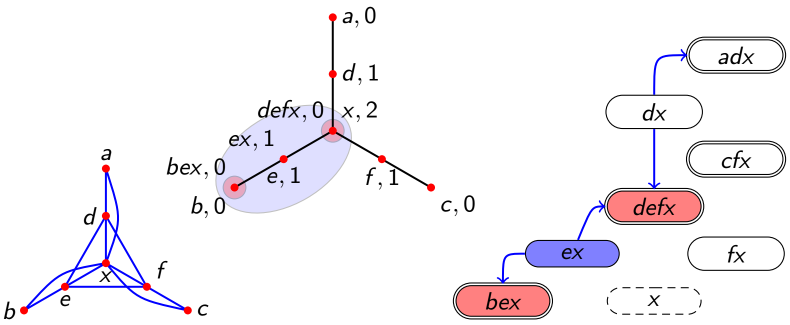

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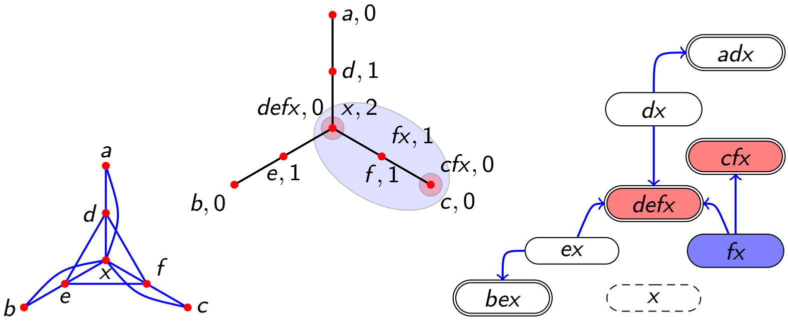

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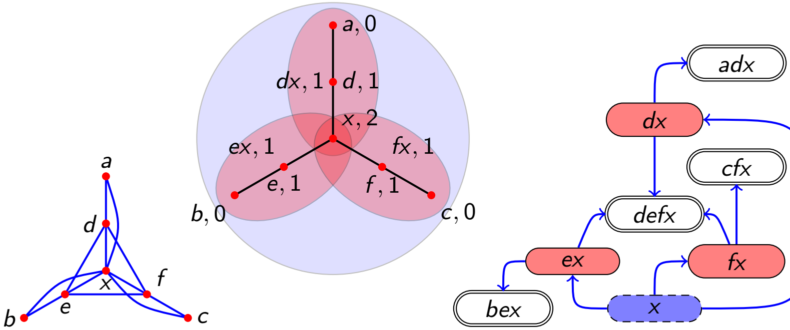
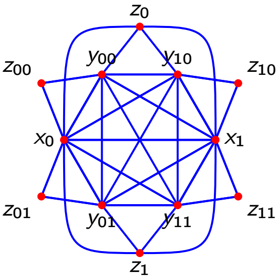


Examine the Counter Example



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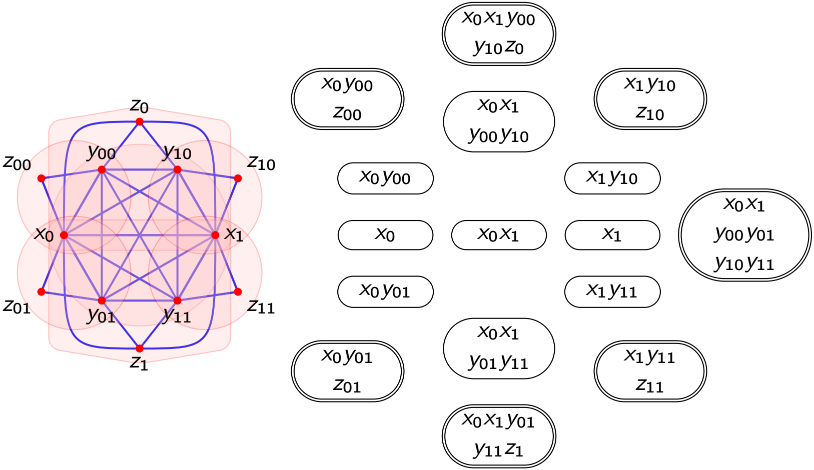

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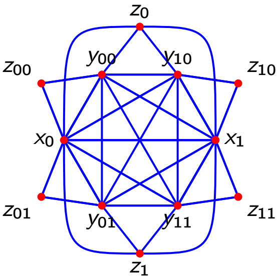

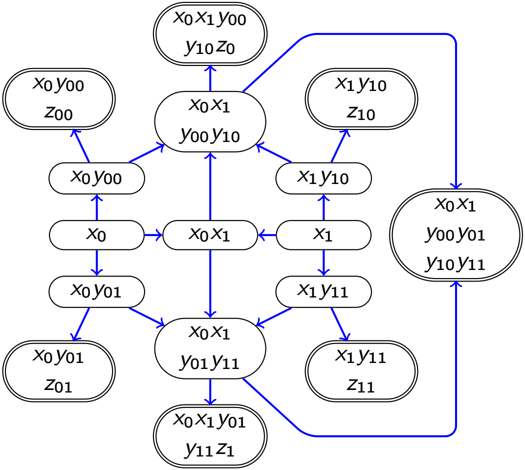

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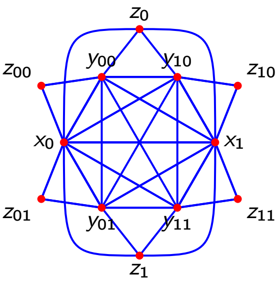
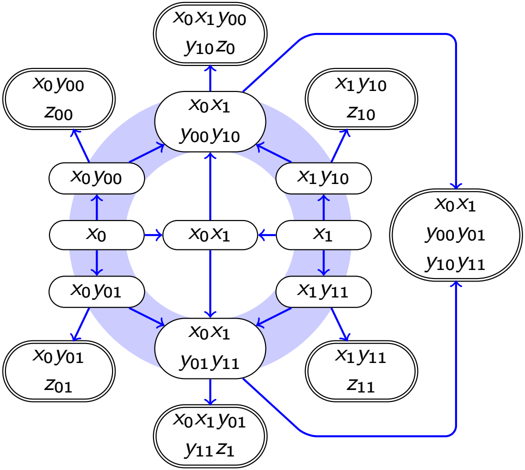


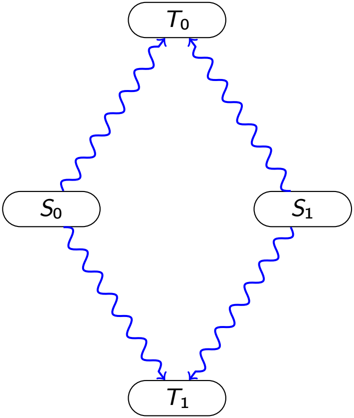


Small Bad Cycles in the Clique Arrangement

Bad 2-Cycle

- start nodes S_0, S_1
- terminal nodes T_0, T_1,
- such that S_0 and S_1 reach T_0 and T_1
- without seeing a node X with

$$S_0 \cup S_1 \subseteq X \subseteq T_0 \cap T_1$$



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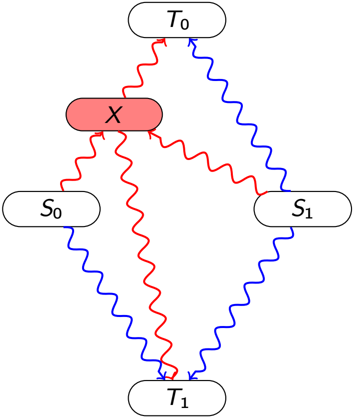
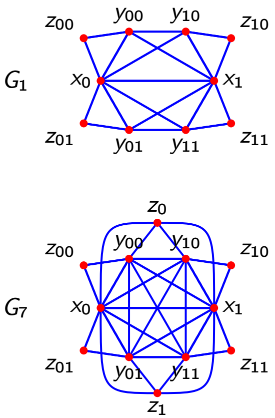
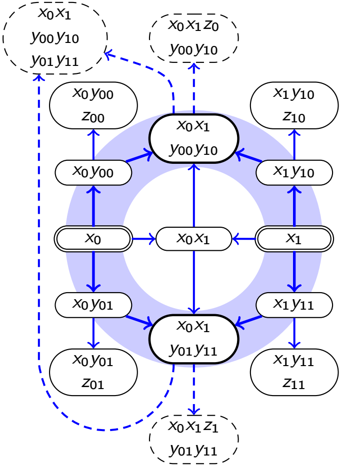


Seven more Graphs with Bad 2-Cycles





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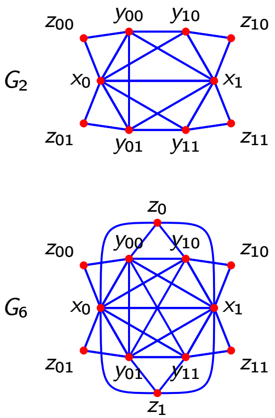

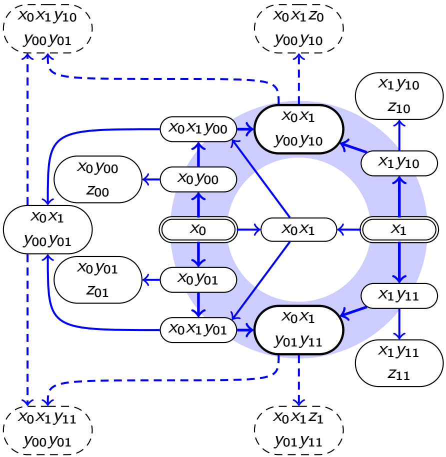

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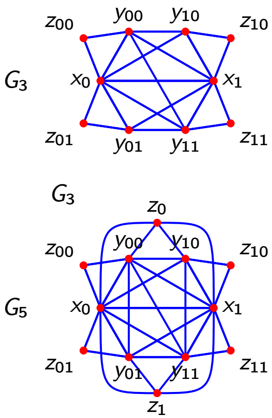
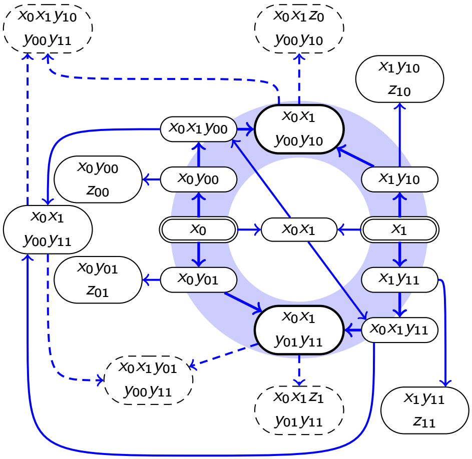
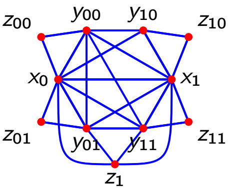
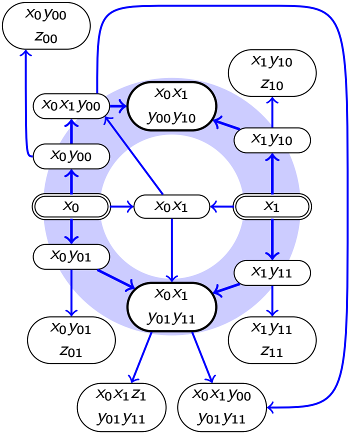




Seven more Graphs with Bad 2-Cycles

G_4

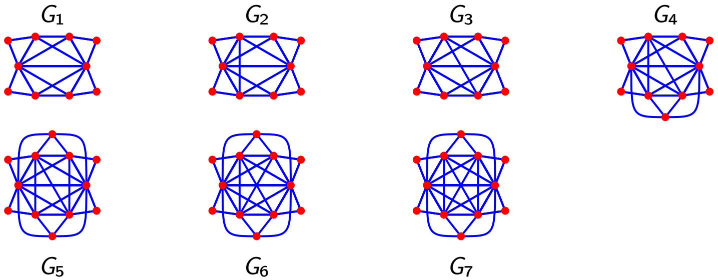




Characterization of Graphs with Bad 2-cycles

Theorem

The clique arrangement of a strongly chordal graph G contains a bad 2-cycle if and only if G has an induced $G_1, \ldots, G_7$.

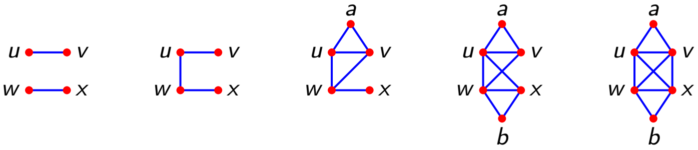


Critical Edges

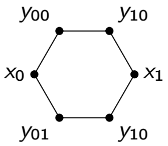
Lemma

Every Leaf Root of a graph $G = (V, E)$ with $uv, wx \in E$ has a u-v-path disjoint from the w-x-path, if

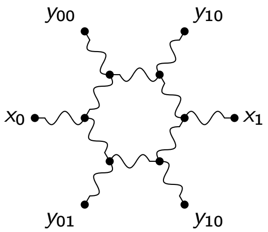
1. $|\{uw, ux, vw, vx\} \cap E| \leq 1$,
2. $\exists a$ *with* $u, v \in N(a)$ *and* $w, x \notin N[a]$ *and* $N(u) \cap \{w, x\} \leq 1$ *and* $N(v) \cap \{w, x\} \leq 1$,
3. $\exists a \neq b$ *with* $u, v \in N(a) \setminus N[b]$ *and* $w, x \in N(b) \setminus N[a]$.



New Counter Examples for Leaf Powers



Leaf Root:



Theorem

$G_1, \ldots, G_7$ are not leaf powers.

Corollary

If G is a leaf power then the clique arrangement of G is free of bad k-cycles, $k \geq 2$.

Conclusions and Future Research

- Better understanding of leaf power graph structure.
- More insights into acyclicity of clique arrangements.
- More strongly chordal graphs that are not a leaf power.
- Possible way to finally solve leaf power recognition.

Thanks for your attention!